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Spectral analysis and multidimensional stability of traveling waves for nonlocal Allen–Cahn equation

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Abstract

In this paper, the spectrum of linearized operator about a traveling wave for the nonlocal Allen–Cahn equation is estimated and the result is applied to study multidimensional stability of planar waves.

1. Introduction

Much attention has been paid to traveling wave solutions of the nonlocal Allen–Cahn equation

$$u_t = J * u - u + f(u). \tag{1.1}$$

Here $J \in C^1(\mathbb{R})$ is a nonnegative function with $\int_{\mathbb{R}} J(y)\,dy = 1$; $J * u = \int_{\mathbb{R}} J(x-y)u(y)\,dy$ is the convolution of J and u; f is a smooth function with three zeros, ± 1 and $a \in (-1, 1)$ satisfying $f'(\pm 1) < 0$ and $f'(a) > 0$. A typical example

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is $f(u) = (u - a)(1 - u^2)$ for some $a \in (-1, 1)$. It is well known that there exists a traveling wave solution of the form $u(x,t) = \phi(x - c_0 t)$ satisfying $\phi(\pm\infty) = \pm 1$, where ϕ is a monotone function and if ϕ is continuous, $c_0 = \int_{-1}^{1} f(u)\,du / \int_{-\infty}^{\infty} (\phi'(z))^2\,dz$; if $c_0 \neq 0$, the traveling wave solution is smooth unique modulo a spatial shift and it is uniformly and asymptotically stable (see [4,5,7]). If the unique speed $c_0 = 0$, the wave may be discontinuous but monotone waves are still unique up to a spatial shift.

Suppose $u(x,t) = \phi(x - c_0 t)$ is a traveling wave solution of (1.1) satisfying $\phi(\pm\infty) = \pm 1$. Let $\xi = x - c_0 t$. Then (ϕ, c_0) satisfies

$$c_0\phi' + J * \phi - \phi + f(\phi) = 0 \quad \text{and} \quad \phi(\pm\infty) = \pm 1 \tag{1.2}$$

for all $\xi \in \mathbb{R}$. The main concern of this paper is the spectrum of the linearized operator of Eq. (1.2) about a traveling wave ϕ. Let $\mathcal{L} = \mathcal{L}(\phi, c_0)$ be the linearized operator about the traveling wave ϕ, defined by

$$\mathcal{L}\psi = c_0\psi' + J * \psi - \psi + f'(\phi)\psi \tag{1.3}$$

for $\psi \in D(\mathcal{L})$, where $D(\mathcal{L}) = H^1(\mathbb{R}) \subset L^2(\mathbb{R})$. We prove that the spectrum of $\mathcal{L}$ consists of 0, the principle eigenvalue due to the translation invariance of the traveling waves, and the rest of it is located in the left half plane bounded away from the imaginary axis.

The immediate consequence of this result is the global exponential stability of traveling wave solutions, which is proved in [7] using the moving plane method, applicable due to a comparison principle which holds for (1.1). As another application, we consider stability of planar traveling wave solutions of the multidimensional nonlocal Allen–Cahn equation

$$u_t = \mathcal{A}u + f(u), \tag{1.4}$$

where $\mathcal{A}u = \sum_{i=1}^{n} \mathcal{A}_i u$ and $\mathcal{A}_i u = \int_{\mathbb{R}} J_i(y)u(x_1, \ldots, x_i - y, \ldots, x_n)\,dy - u(x_1, \ldots, x_n)$, $x \in \mathbb{R}^n$; $J_i \in C^1(\mathbb{R})$ is a nonnegative function with unit integral. It is pointed out in [2,3] that the Laplacian operator $\partial_{xx}u$ in one space dimension may be considered as the first-order approximation of the operator $(J * u - u)$. The operator $\mathcal{A}$ can be considered as a generalized version of the Laplacian operator Δ. The multidimensional stability of Eq. (1.1) with $\mathcal{A}$ replaced by Δ is studied in [10] and [11].

Traveling wave solutions of (1.4) are solutions of the form $\phi(\xi) = \phi(k \cdot x - ct)$ and $\phi(\pm\infty) = \pm 1$, where $k \in S^{n-1}$ is a unit vector. Assume $k = (1, 0, \ldots, 0)$. Then $\phi(k \cdot x - ct) = \phi(x_1 - ct)$ satisfies (1.2) with J replaced by J_1. Therefore the existence of such planar traveling wave solutions can be derived from the one-dimensional case. Our main concern is the multidimensional stability.

The global exponential stability in one space dimension is due to the spectral gap. In the multidimensional case, however, the gap disappears due to the effects of the transverse diffusion along the planar wave front and there may exist continuous spectrum all the way up to zero. In this paper we modify the ideas in [11] to establish the global asymptotic stability for $n \geqslant 4$.

The paper is organized as follows. In Section 2, we will study the spectrum of $\mathcal{L}$. In Section 3, we apply the spectral result to prove the multidimensional stability.

2. Spectrum of $\mathcal{L}$

In this section we study the linearized operator $\mathcal{L}$ about a traveling wave solution (ϕ, c_0) satisfying (1.2). Let $L^2(\mathbb{R}^n)$ denote the square integrable functions on $\mathbb{R}^n$ with inner product $(\cdot\,,\cdot)$. The norm induced by the inner product is denoted by $\|\cdot\|$. Denote $H^m(\mathbb{R}^n)$ the Sobolev space of all functions which have weak derivatives up to order m which are square integrable. We use $\|u\|_{H^m(\mathbb{R}^n)} = \{\sum_{|\alpha|\leqslant m} \|\partial^\alpha u\|^2\}^{1/2}$ to denote the norm in $H^m(\mathbb{R}^n)$. We consider the operators $\mathcal{L}$ defined by (1.3) on $L^2(\mathbb{R})$ with domain $D(\mathcal{L}) = H^1(\mathbb{R})$.

Let us define a normal point for an operator $\mathcal{L}$ on a Banach space to be any complex number which is in the resolvent set $\rho(\mathcal{L})$ or is an isolated eigenvalue of $\mathcal{L}$ of finite multiplicity. The complement of the set of normal points is called the essential spectrum of $\mathcal{L}$ and is denoted by $\sigma_{\mathrm{ess}}(\mathcal{L})$. The following is the main result of this section.

Theorem 2.1. *If $c_0 \neq 0$, then*

(i) $\{\lambda\colon\ \mathrm{Re}\,\lambda \geqslant 0,\ \lambda \neq 0\} \subset \rho(\mathcal{L})\}$;
(ii) 0 *is an algebraically simple eigenvalue with a positive eigenfunction* ϕ';
(iii) *there exists* $\gamma_0 > 0$ *such that* $\sigma_{\mathrm{ess}}(\mathcal{L}) \subset \{\lambda\colon\ \mathrm{Re}\,\lambda < -\gamma_0\}$.

An immediate consequence of this is (see [9])

Corollary 2.2. (i) *If $c_0 \neq 0$, the traveling wave solution $\phi(x - c_0 t)$ is globally exponentially stable in $L^2(\mathbb{R})$.*

(ii) *There exist positive constants γ and C such that*

$$\|e^{\mathcal{L}t} u\| \leqslant C e^{-\gamma t} \|u\|.$$

for all u in the range of $\mathcal{L}$.

Remark 2.1. (i) Notice that when $c_0 = 0$, there may exist discontinuous traveling wave (standing wave) solutions with some jumps. In this case ϕ' has δ-function type discontinuities and so ϕ' is not in $L^2(\mathbb{R})$.

(ii) If $c_0 \neq 0$, since $\phi' > 0$, from $|c_0|(\phi', \phi') = |(\phi - J * \phi - f(\phi), \phi')| \leqslant 2(2 + \max_{u\in[-1,1]} |f(u)|)$, we deduce that $\phi' \in L^2(\mathbb{R})$. Hence, if $f \in C^m$, it follows that $\phi' \in H^m(\mathbb{R}^n)$ from induction arguments.

Therefore, in the sequel, we assume that $c_0 \neq 0$. We divide the proof of the theorem into several lemmas. Let ζ be a smooth function satisfying $\zeta(x) = 0$ for $x \leqslant -1$, $\zeta(x) = 1$ for $x \geqslant 1$ and $\zeta'(x) > 0$ for $x \in (-1, 1)$, and let $s(x) = f'(-1)(1 - \zeta(x)) + f'(1)\zeta(x)$. Define an operator $\mathcal{L}_0$ on $L^2(\mathbb{R})$ with domain $D(\mathcal{L}_0) = H^1(\mathbb{R})$ by $\mathcal{L}_0\psi = c_0\psi' + J * \psi - \psi + s(x)\psi$ for $\psi \in D(\mathcal{L}_0)$. Then we have

Lemma 2.3. $\{\lambda\colon \operatorname{Re}\lambda > \max\{f'(-1), f'(1)\}\} \subset \rho(\mathcal{L}_0)$.

Proof. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \max\{f'(-1), f'(1)\}$. Since $|\hat{J}(\xi)| \leqslant \int_{\mathbb{R}} J(x)\,dx = 1$, from the definition of Fourier transform, we have $\operatorname{Re}(u - J * u, u) = \int_{\mathbb{R}}(1 - \operatorname{Re}\hat{J}(\xi))|\hat{u}(\xi)|^2\,d\xi \geqslant 0$ for all $u \in L^2(\mathbb{R})$. Therefore, if $(\lambda - \mathcal{L}_0)u = 0$ for some $u \in D(\mathcal{L}_0)$, then

$$\begin{aligned}
0 &= \operatorname{Re}\bigl((\lambda - \mathcal{L}_0)u, u\bigr)\\
&= \operatorname{Re}\bigl(\bigl(\lambda - s(x)\bigr)u, u\bigr) + \operatorname{Re}(u - J * u, u) - c_0 \operatorname{Re}(u', u)\\
&\geqslant \bigl(\operatorname{Re}\lambda - \max\bigl\{f'(-1), f'(1)\bigr\}\bigr)\|u\|^2,
\end{aligned}$$

where we have used the fact $2\operatorname{Re}(u', u) = \int_{\mathbb{R}}(|u|^2)'\,dx = 0$. Therefore, $u = 0$ and $\lambda - \mathcal{L}_0$ is one-to-one.

To prove $\lambda - \mathcal{L}_0$ is surjective, choose a constant ρ satisfying $\max\{f'(-1), f'(1)\} < \rho < \operatorname{Re}\lambda$ and consider two operators $\mathcal{L}_1$ and $\mathcal{L}_2$ on $L^2(\mathbb{R})$ defined by $\mathcal{L}_1\psi = (\lambda - \rho)\psi + \psi - J * \psi - c_0\psi'$ for $\psi \in D(\mathcal{L}_1) = H^1(\mathbb{R})$, and $\mathcal{L}_2\psi = (\rho - s(x))\psi$ for $\psi \in D(\mathcal{L}_2) = L^2(\mathbb{R})$, respectively. It is easy to see that $\operatorname{Re}(\mathcal{L}_1\psi, \psi) \geqslant (\operatorname{Re}\lambda - \rho)\|\psi\|^2$. Therefore $\mathcal{L}_1$ is a monotone operator. Moreover, for any $g \in L^2(\mathbb{R})$, $u(x) = \mathcal{F}^{-1}((\lambda - \rho + 1 - \hat{J}(\xi) + ic_0\xi)^{-1}\hat{g}(\xi))$ in the solution for $\mathcal{L}_1 u = g$, where we use $\mathcal{F}^{-1}(u)$ to denote the inverse Fourier transform. Moreover, $\|u\| = \|\hat{u}\| = \|(\lambda - \rho + 1 - \hat{J}(\xi) + ic_0\xi)^{-1}\hat{g}(\xi)\| \leqslant (\operatorname{Re}\lambda - \rho)^{-1}\|g\|$. Therefore $u \in H^1(\mathbb{R})$ from the equation and $\mathcal{L}_1$ is hypermaximal monotone. Obviously, $\mathcal{L}_2$ is a homeomorphism on $L^2(\mathbb{R})$ and $\mathcal{L}_2$ is also monotone. Therefore $\lambda - \mathcal{L}_0 = \mathcal{L}_1 + \mathcal{L}_2$ is hypermaximal monotone and, in particular, it is surjective (see [8]). Therefore $(\lambda - \mathcal{L}_0)^{-1}$ exists. It is easy to see that $(\operatorname{Re}\lambda - \rho)\|(\lambda - \mathcal{L}_0)^{-1} f\| \leqslant C\|f\|$ and so $\lambda \in \rho(\mathcal{L}_0)$. $\square$

Suppose $q \in L^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} q(x) = 0$ and define a bounded linear operator B on $L^2(\mathbb{R})$ by $Bu = qu$ for $u \in L^2(\mathbb{R})$. Then we have

Lemma 2.4. *Let $\lambda \in \rho(\mathcal{L}_0)$. Then $B(\lambda - \mathcal{L}_0)^{-1}$ is a compact operator on $L^2(\mathbb{R})$.*

Proof. Let C be a bounded closed set in $L^2(\mathbb{R})$. Since $\|(\lambda - \mathcal{L}_0)^{-1} f\|_{H^m(\mathbb{R})} \leqslant C\|f\|$ and $m \geqslant 1$, $(\lambda - \mathcal{L}_0)^{-1}C$ is bounded in $H^1(\mathbb{R})$ and therefore it is compact in $L^2([-n, n])$ for all $n > 0$. We can use a diagonal argument to get a sequence $\{u_k\} \subset (\lambda - \mathcal{L}_0)^{-1}C$ such that u_k converges to some u in $L^2([-n, n])$ for each fixed n. Since $\lim_{x\to\pm\infty} q(x) = 0$, it easily follows that Bu_k converges to Bu in $L^2(\mathbb{R})$. $\square$

Now take $q(x) = f'(\phi(x)) - s(x)$ and note that $\mathcal{L} = \mathcal{L}_0 + B$. By Theorem A.1 in [9, p. 136], we deduce that the half plane $\{\lambda \mid \operatorname{Re}\lambda > \max\{f'(-1), f'(1)\}\}$ consists entirely of normal points of $\mathcal{L}$. This establishes the claim (iii) of Theorem 2.1. Parts (i) and (ii) will follow from Lemmas 2.5–2.7 below.

Lemma 2.5. $\{\lambda\colon \operatorname{Re}\lambda \geqslant 0,\ \lambda \neq 0\} \subset \rho(\mathcal{L})$.

Proof. First notice that every eigenfunction is at least C^1, since $c_0 \neq 0$. Suppose $\lambda = \alpha + i\beta$ satisfying $\alpha \geqslant 0$ and $\beta \neq 0$ is an eigenvalue with eigenfunction $u = u^1(x) + iu^2(x) \neq 0$. Consider the Cauchy problem

$$v_t = \mathcal{L}v - \alpha v, \tag{2.1}$$

$$v(x,0) = u^1(x). \tag{2.2}$$

It has a solution $v(x,t) = u^1(x)\cos\beta t - u^2(x)\sin\beta t$.

Note that $v(x,t) \leqslant |u(x)|$ for all $x \in \mathbb{R}$ and $t \geqslant 0$. We claim that there is a $\tau > 0$ such that $v(x,t) \leqslant \tau\phi'(x)$ for all $x \in \mathbb{R}$ and $t \geqslant 0$. To prove this claim, let θ_0 be a constant satisfying $0 < \theta_0 < \min\{-f'(-1), -f'(1)\}$ and choose M large enough such that $f'(\phi(x)) \leqslant -\theta_0$ for all $|x| \geqslant M$ and such that $|u|$ is positive at some point $x \in [-M, M]$. Since $\phi'(x) > 0$, there exists a constant $\tau > 0$ such that $|u(x)| \leqslant \tau\phi'(x)$ for $|x| \leqslant M$. We prove that the claim holds with this choice of τ. Since $\lim_{\pm x\to\infty} u(x) = 0$, there exists a constant $\epsilon > 0$ such that $v(x,t) \leqslant \tau\phi'(x) + \epsilon$ for all x and $t \geqslant 0$. Let $\epsilon_0 = \inf\{\epsilon\colon v(x,t) \leqslant \tau\phi'(x) + \epsilon,$ for all $x \in \mathbb{R}$ and $t \geqslant 0\}$. We prove that $\epsilon_0 = 0$. Consider the function $w(x,t) = \tau\phi'(x) + \epsilon_0 e^{-\theta_0 t}$. We have

$$\begin{aligned} w_t - \mathcal{L}w + \alpha w &= \epsilon_0\bigl(-\theta_0 - f'(\phi)\bigr)e^{-\theta_0 t} + \alpha w \\ &\geqslant \epsilon_0\bigl(-\theta_0 - f'(\phi)\bigr)e^{-\theta_0 t} \geqslant 0 \end{aligned} \tag{2.3}$$

for all $|x| > M$ and $t > 0$. Therefore w is a super-solution of (2.1) on $|x| > M$. Notice that $w(x,t) \geqslant v(x,t)$ for $|x| \leqslant M$ and $t > 0$ and $w(x,0) \geqslant v(x,0)$ for all x. The comparison principle (see [1,5]) yields $w(x,t) \geqslant v(x,t)$ for all x and $t > 0$. Therefore $v(x,t) = v(x, t + 2n\pi/\beta) \leqslant \tau\phi'(x) + \epsilon_0 e^{-\theta_0(t+2n\pi/\beta)}$, for all $n \in Z^+$, $x \in \mathbb{R}$ and $t > 0$. Letting $n \to \infty$, we get $v(x,t) \leqslant \tau\phi'(x)$ for all x. Therefore $\epsilon_0 = 0$ and the claim is proved.

Clearly τ_0 can be chosen such that $|u(x)| \leqslant \tau_0\phi'(x)$ for all $|x| \leqslant M$ and there is a point $x_0 \in [-M, M]$ such that $|u(x_0)| = \tau_0\phi'(x_0)$. By the strong comparison principle, we deduce that $v(x,t) < \tau_0\phi'(x)$ for all x and $t > 0$. If we choose t such that $u(x_0)/|u(x_0)| = e^{-i\beta t}$, then $v(x_0,t) = |u(x_0)| = \tau_0\phi'(x_0) > v(x_0,t)$, which is a contradiction. Therefore $u(x) = 0$ for all x and λ is not an eigenvalue.

Now assume that $\lambda > 0$ is an eigenvalue with an eigenfunction $u(x)$. Without loss of generality, we may assume u is real and there is a point where u is positive. Let M be chosen as before, choose τ large enough such that $w(x;\tau) \equiv \tau\phi'(x) - u(x) \geqslant 0$ for $|x| \leqslant M$. Then $w(x;\tau) \geqslant 0$ for all x. In fact, if not, there exists a point x_0 with $|x_0| > M$ such that $w(x_0;\tau) = \min_{x\in\mathbb{R}}\{w(x;\tau)\} < 0$. Then,

$$0 > -\lambda\tau\phi'(x_0) = \mathcal{L}w - \lambda w \geqslant -\lambda w(x_0;\tau) > 0$$

which is a contradiction. Therefore, there is a constant $\tau > 0$ such that $w(x;\tau) \geqslant 0$. Let $\tau_0 = \inf\{\tau\colon\ w(x;\tau) \geqslant 0, \text{ for all } x \in \mathbb{R}\}$. We claim that $\tau_0 = 0$. In fact, if $\tau_0 > 0$ and $\tau_n < \tau_0$ is a sequence such that $\lim_{n\to\infty}\tau_n \to \tau_0$, for each n, $w(x,\tau_n)$ has a negative value at some point. Let x_n be the minimum point of $w(x,\tau_n)$, that is, $w(x_n,\tau_n) = \min_{x\in\mathbb{R}} w(x,\tau_n) < 0$. Then, from the equation $\mathcal{L}w - \lambda w = -\lambda\tau_n\phi'$, we deduce that $|x_n| \leqslant M$. Therefore, there exists a subsequence of x_n, which we label the same way, such that $\lim_{n\to\infty} x_n = \bar{x}$ for some $\bar{x}$. Taking limits along x_n and τ_n in the equation $\mathcal{L}w - \lambda w = -\lambda\tau_n\phi'$, we reach a contradiction since $w(\bar{x};\tau_0) = 0$. Therefore $\tau_0 = 0$ and $u \leqslant 0$, which contradicts the assumption. We conclude that $\lambda > 0$ is not an eigenvalue. □

Lemma 2.6. 0 *is a simple eigenvalue of* $\mathcal{L}$ *with positive eigenfunction* ϕ'.

Proof. We know that ϕ' is an eigenfunction corresponding to eigenvalue 0. Let $\psi \in H^1(\mathbb{R})$ be another eigenfunction. Without loss of generality, we may assume $\psi(x) > 0$ at some point. As in the proof of Lemma 2.5, one finds that there exists a constant $\tau > 0$ such that $\psi(x) \leqslant \tau\phi'(x)$ for all x. Let $\tau_0 = \inf\{\tau\colon\ \tau\phi'(x) \geqslant u(x)$ for all $x\}$. Again, following the proof of Lemma 2.5, one can show that $\tau_0\phi'(x) = \psi(x)$ for all x. □

Let

$$\mathcal{L}^*\psi = -c_0\psi' + J * \psi - \psi + f'(\phi)\psi \tag{2.4}$$

with domain $D(\mathcal{L}^*) = H^1(\mathbb{R})$ be the formal adjoint operator of $\mathcal{L}$. Then we have

Lemma 2.7. *The equation* $\mathcal{L}^* v = 0$ *on* $H^1(\mathbb{R})$ *has a positive solution. It is unique modulo a constant multiple.*

Proof. Since 0 is a simple eigenvalue of $\mathcal{L}$, by the Fredholm theory, $\mathcal{L}^* v = 0$ has a nonzero solution v and it is unique modulo a constant multiple. For any solution v of $\mathcal{L}^* v = 0$ in $H^1(\mathbb{R})$, $v(x)$ must have the same sign for all x. For otherwise, if v changes sign, we can find a continuous function $f \in L^2(\mathbb{R})$ such that $f(x) > 0$ for all x and $(f,v) = 0$. Let u be a solution of $\mathcal{L}u = f$, which exists by Fredholm theory. Choose τ such that $\tau\phi'(x) \geqslant u(x)$ and $\tau\phi'(x_0) = u(x_0)$ for some x_0. Then $\mathcal{L}(\tau\phi' - u)(x_0) \geqslant 0$, which contradicts the fact $\mathcal{L}u(x_0) = f(x_0) > 0$. □

3. Multidimensional stability

In this section, we apply results obtained in Section 2 to study multidimensional stability of planar traveling waves. Without loss of generality, we may assume $J_1 = J_2 = \cdots = J_n = J$ and $J \in C^1(\mathbb{R})$ is a nonnegative function with $\int_{\mathbb{R}} J(y)\,dy = 1$. We assume furthermore that J is an even function; that is, $J(-x) = J(x)$ for all x and $xJ(x)$, $x^2 J(x) \in L^1(R)$. For simplicity, we assume $f(u) = (u-a)(1-u^2)$ for some $a \in (-1,1)$ with $a \neq 0$. From the result in the one-dimensional case, we deduce that Eq. (1.4) has a traveling wave solution of the form $u(t,x) = \phi(x_1 - c_0 t)$ satisfying $\phi(\pm\infty) = \pm 1$, where ϕ is smooth and $c_0 \neq 0$. It can be seen from (1.2) that $\phi \in H^m(\mathbb{R})$ for all $m \in \mathbb{Z}^+$. Notice that a traveling wave solution is translation invariant. We fix one such traveling wave solution (ϕ, c_0) and study the asymptotic stability of the planar traveling wave $\phi(x_1 - c_0 t)$ under multidimensional perturbation. We have the following result.

Theorem 3.1. *Let $u(t,x)$ be the solution of* (1.4) *with initial condition $u(0,x) = \phi(x_1) + v_0(x)$. Assume that $v_0 \in H^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $n \geqslant 4$ and integer $m \geqslant n+1$. Then there exist constants $\epsilon > 0$ and $C > 0$ such that if $\|v_0\|_{H^m(\mathbb{R}^n)} + \|v_0\|_{L^1(\mathbb{R}^n)} < \epsilon$, then*

$$\big\|u(t,x) - \phi(x_1 - c_0 t)\big\|_{H^m(\mathbb{R}^n)} \leqslant C(1+t)^{-(n-1)/4} \tag{3.1}$$

for $t \geqslant 0$.

To prove the theorem, we first introduce some notations and lemmas. Let $\mathcal{L}$ be defined as in (1.3) and the formal adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ be defined as in (2.4). From Lemma 2.7, 0 is an simple eigenvalue of $\mathcal{L}^*$ with a positive eigenfunction ϕ^*. We normalize ϕ^* so that $\int_{\mathbb{R}} \phi'(x)\phi^*(x)\,dx = 1$ and let $\bar{P}v = \int_{\mathbb{R}} v\phi^*\,d\xi$ for $v(\xi, y) \in L^2(\mathbb{R}^n)$. Define two projection operators on $L^2(\mathbb{R}^n)$ by

$$Pv = \phi' \bar{P} v \tag{3.2}$$

and

$$Qv = v^{\perp} = v - Pv. \tag{3.3}$$

Then, $(Qv, \phi^*) = 0$.

Suppose $u(0,x) = \phi(x_1) + v_0(x)$, where $v_0(x)$ is small in some sense to be made clear. Write the solution $u(t,x)$ of (1.4) with initial value $u(0,x)$ as

$$u(t,x) = \phi(x_1 - c_0 t) + v(t,x). \tag{3.4}$$

Then $v(t,x)$ satisfies

$$v_t = \mathcal{A}v + f(\phi + v) - f(\phi), \tag{3.5}$$

$$v(0,x) = v_0(x). \tag{3.6}$$

If we work in the traveling wave frame and let $\xi = x_1 - c_0 t$ and $y = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, then (3.5) becomes

$$v_t = \mathcal{L}v + \mathcal{B}v + R(v, \phi), \tag{3.7}$$

where $\mathcal{B} = \sum_{i=2}^{n} \mathcal{A}_i$ and $R(v, \phi) = f(\phi + v) - f(\phi) - f'(\phi)v$. To solve (3.7), we decompose v in (3.7) as $v = Pv + Qv$ and apply the projection operators P and Q on both sides of (3.7). Then (3.7) becomes

$$v_t^{\perp} = \mathcal{L}v^{\perp} + \mathcal{B}v^{\perp} + QR(v, \phi), \tag{3.8}$$

$$p_t = \mathcal{B}p + \bar{P}R(v, \phi), \tag{3.9}$$

where $p = p(t, y) = \bar{P}v$. We are going to solve (3.8) and (3.9) with the initial data $v^{\perp}|_{t=0} = Qv_0$ and $p|_{t=0} = \bar{P}v_0$. First we need the following lemma.

Lemma 3.2. *Let u be the solution of the initial value problem*

$$u_t = \mathcal{A}u, \tag{3.10}$$

$$u|_{t=0} = u_0, \tag{3.11}$$

where $u_0 \in H^m(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ for some $N \geqslant 1$. Then there exist positive constants β and $C = C(\beta)$ such that

$$\begin{aligned} \left\| \partial_x^{\alpha} u \right\|_{L^2(\mathbb{R}^N)} \leqslant{}& e^{-\beta t} \left\| \partial_x^{\alpha} u_0 \right\|_{L^2(\mathbb{R}^N)} \\ &+ C(1+t)^{-N/4-|\alpha|/2} \|u_0\|_{L^1(\mathbb{R}^N)}, \end{aligned} \tag{3.12}$$

for all $\alpha \in \mathbb{Z}_+^N$ with $|\alpha| \leqslant m$.

Proof. By the assumptions for J, we know that the Fourier transform $\hat{J}$ is well defined. Moreover, $\hat{J}(0) = \int_{\mathbb{R}^1} J(x)\,dx = 1$ and $\hat{J}'(0) = -i \int_{\mathbb{R}^1} xJ(x)\,dx = 0$ since $J(x)$ is even. By the continuity of $\hat{J}''(\eta)$ and the fact that $\hat{J}''(0) = -\int_{\mathbb{R}^1} x^2 J(x)\,dx < 0$, there exist constants $\delta > 0$ and $\theta > 0$ such that $\hat{J}''(\eta) \leqslant -\theta < 0$ for all η with $|\eta| \leqslant \delta$. Therefore,

$$\hat{J}(\eta) - 1 = \int_0^1 \hat{J}''(t\eta)(1-t)\,dt\,\eta^2 \leqslant -\frac{\theta}{2}\eta^2 \tag{3.13}$$

for $|\eta| \leqslant \delta$. On the other hand, since $\hat{J}(\eta) = 2\int_0^{\infty} J(x)\cos(x\eta)\,dx$, we see that $\lim_{|\eta| \to \infty} \hat{J}(\eta) = 0$. Therefore there exists a positive constant $C(\delta)$ such that

$$1 - \hat{J}(\eta) \geqslant C(\delta) \tag{3.14}$$

for all $|\eta| \geqslant \delta/\sqrt{N}$.

First let us prove the lemma for $\alpha = 0$. Taking Fourier transforms in (3.10) and (3.11), we get

$$\hat{u}_t(t,\eta) = \sum_{k=1}^{N} \bigl(\hat{J}(\eta_k) - 1\bigr)\hat{u}(t,\eta) \quad \text{and} \quad \hat{u}(0,\eta) = \hat{u}_0(\eta). \tag{3.15}$$

It follows that

$$\hat{u}_t(t,\eta) = \exp\left(\sum_{k=1}^{N} \bigl(\hat{J}(\eta_k) - 1\bigr)t\right)\hat{u}_0(\eta). \tag{3.16}$$

Therefore

$$\begin{aligned}\|u\|_{L^2}^2 = \|\hat{u}\|_{L^2}^2 &= \int_{\mathbb{R}^N} \exp\left(2\sum_{k=1}^{N} \bigl(\hat{J}(\eta_k) - 1\bigr)t\right)\bigl|\hat{u}_0(\eta)\bigr|^2\,d\eta \\ &= \mathrm{I} + \mathrm{II},\end{aligned} \tag{3.17}$$

where $\mathrm{I} = \int_{|\eta|\geqslant\delta} k(\eta)\,d\eta$, $\mathrm{II} = \int_{|\eta|\leqslant\delta} k(\eta)\,d\eta$, $k(\eta) = \exp(2\sum_{k=1}^{N}(\hat{J}(\eta_k) - 1)t) \times |\hat{u}_0(\eta)|^2$ and δ is chosen as above. For I, we have

$$\begin{aligned}\mathrm{I} &= \int_{|\eta|\geqslant\delta} \exp\left(2\sum_{k=1}^{N} \bigl(\hat{J}(\eta_k) - 1\bigr)t\right)\bigl|\hat{u}_0(\eta)\bigr|^2\,d\eta \\ &\leqslant e^{-2C(\delta)t} \int_{|\eta|\geqslant\delta} \bigl|\hat{u}_0(\eta)\bigr|^2\,d\eta \leqslant e^{-2C(\delta)t}\|u_0\|^2,\end{aligned} \tag{3.18}$$

where we have used (3.14).

For II, we have

$$\begin{aligned}\mathrm{II} &= \int_{|\eta|\leqslant\delta} \exp\left(2\sum_{k=1}^{N} \bigl(\hat{J}(\eta_k) - 1\bigr)t\right)\bigl|\hat{u}_0(\eta)\bigr|^2\,d\eta \\ &\leqslant \bigl\|\hat{u}_0(\eta)\bigr\|_{L^\infty(\mathbb{R}^N)}^2 \int_{|\eta|\leqslant\delta} \exp\left(2\sum_{k=1}^{N} \bigl(\hat{J}(\eta_k) - 1\bigr)t\right)d\eta \\ &\leqslant \bigl\|u_0(\eta)\bigr\|_{L^1(\mathbb{R}^N)}^2 \left[\int_{-\delta}^{\delta} e^{2(\hat{J}(\eta)-1)t}\,d\eta\right]^N \\ &\leqslant \bigl\|u_0(\eta)\bigr\|_{L^1(\mathbb{R}^N)}^2 \left[\int_{-\delta}^{\delta} e^{-\theta\eta^2 t}\,d\eta\right]^N,\end{aligned} \tag{3.19}$$

where we have used (3.13). It is easy to see that there exists a constant $C = C(\theta)$ such that $\int_{-\delta}^{\delta} e^{-\theta\eta^2 t}\,d\eta \leqslant C(1+t)^{-1/2}$. Estimate (3.12) follows from (3.18) and (3.19) for the case $\alpha = 0$. For the general case, by taking Fourier transforms, we have

$$\|\partial_x^\alpha u\|^2_{L^2(\mathbb{R}^N)} = \|\eta^\alpha \hat{u}\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \exp\left(2\sum_{k=1}^{N}\big(\hat{J}(\eta_k)-1\big)t\right)\left|\eta^\alpha \hat{u}_0(\eta)\right|^2 d\eta,$$

and the required estimate (3.12) can be obtained similarly. □

Lemma 3.3. *Suppose* $h(t,\xi,y) \in H^m(\mathbb{R}^n)$*, for all* $t > 0$*, and satisfies* $\sup_{t\geqslant 0}(1+t)^\alpha \|h\|_{H^m}(t) < \infty$*, for some constant* $\alpha > 0$*. If* w *is the unique solution of the following initial value problem*:

$$w_t = \mathcal{L}w + \mathcal{B}w + Qh, \tag{3.20}$$

$$w|_{t=0} = 0, \tag{3.21}$$

then $\sup_{t\geqslant 0}(1+t)^\alpha \|w\|_{H^m}(t) \leqslant C \sup_{t\geqslant 0}(1+t)^\alpha \|h\|_{H^m}(t)$*, where* C *is some constant depending only on* ϕ*,* ϕ^**,* m *and* α*.*

Proof. First, the local existence and uniqueness of the solution w follows from the classical semigroup theory since $\mathcal{A}$ is a bounded operator on $H^m(\mathbb{R}^n)$. Multiplying both sides of (3.20) by ϕ^* and integrating over ξ, we get

$$(\bar{P}w)_t = (\mathcal{L}w,\phi^*) + (\mathcal{B}w,\phi^*) + (Qh,\phi^*) = \mathcal{B}(\bar{P}w). \tag{3.22}$$

Therefore $\bar{P}w = 0$ for all $t > 0$ and all $y \in \mathbb{R}^{n-1}$, since $\bar{P}w|_{t=0} = 0$. Therefore, $Pw = 0$. Since $\widehat{\mathcal{B}w} = \sum_{i=2}^n (\hat{J}(\eta_i) - 1)\hat{w}$, taking the Fourier transform for $y = (x_2,\dots,x_n)$ on both sides of (3.20) yields

$$\hat{w}_t = \mathcal{L}\hat{w} + \sum_{i=2}^{n}\big(\hat{J}(\eta_i)-1\big)\hat{w} + \hat{Q}h, \tag{3.23}$$

where $\hat{w} = \hat{w}(t,\xi,\eta')$ and $\eta' = (\eta_2,\dots,\eta_n)$. Solving (3.23) with initial data $\hat{w}|_{t=0} = 0$, we get

$$\hat{w} = \int_0^t e^{(t-s)(\mathcal{L}+\sum_{i=2}^n(\hat{J}(\eta_i)-1))} Q\hat{h}\,ds,$$

where we have used the fact $\hat{Q}h = Q\hat{h}$. By Corollary 2.2, there exist positive constants γ and C such that $\|e^{\mathcal{L}t}u\|_{L^2(\mathbb{R})} \leqslant Ce^{-\gamma t}\|u\|_{L^2(\mathbb{R})}$ for all $u \in \{\phi^*\}^\perp \subset L^2(\mathbb{R})$. Since $\hat{J}(\eta_i) \leqslant 1$, we have

$$\|\hat{w}\|_{L^2_\xi} \leqslant C\int_0^t e^{-\gamma(t-s)}\|Q\hat{h}\|_{L^2_\xi}(s)\,ds.$$

By the Minkowski inequality and the conditions on h, we have

$$\begin{aligned}\|w\|_{L^2(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^{n-1}} \|\hat{w}(\xi,\eta)\|^2_{L^2(\mathbb{R})}\,d\eta\right)^{1/2}\\ &\leqslant C\left\|\int_0^t e^{-\gamma(t-s)}\|Q\hat{h}\|_{L^2_\xi}(s)\,ds\right\|_{L^2(\mathbb{R}^{n-1})}\\ &\leqslant C\int_0^t e^{-\gamma(t-s)}\|Qh\|_{L^2(\mathbb{R}^n)}\,ds\\ &\leqslant C\sup_{\tau\geqslant 0}(1+\tau)^\alpha\|h\|_{L^2(\mathbb{R}^n)}(\tau)\int_0^t e^{-\gamma(t-s)}(1+s)^{-\alpha}\,ds\\ &\leqslant C(1+t)^{-\alpha}\sup_{\tau\geqslant 0}(1+\tau)^\alpha\|h\|_{L^2(\mathbb{R}^n)}(\tau).\end{aligned}\tag{3.24}$$

Therefore the lemma holds for $m=0$. Notice that $\partial^k_{y_i}w$ satisfies the same equations (3.20) and (3.21) with Qh replaced by $Q(\partial^k_{y_i}h)$. Therefore we can get estimates for $\|\partial^k_{y_i}w\|_{L^2(\mathbb{R}^n)}$ similarly. Now we need to estimate $\|\partial^k_\xi w\|_{L^2(\mathbb{R}^n)}$. Let $v(t,\xi,y)=\partial_\xi w$. Since $(\mathcal{L}w,\phi^*)_{L^2(\mathbb{R})}=(w,\mathcal{L}^*\phi^*)_{L^2(\mathbb{R})}=0$, we have $c_0(w_\xi,\phi^*)_{L^2(\mathbb{R})}=-(J*w-w+f'(\phi)w,\phi^*)_{L^2(\mathbb{R})}$. Therefore

$$\|Pw_\xi\|_{L^2(\mathbb{R}^n)}\leqslant \left|c_0^{-1}\right|\|\phi'\|_{L^2(\mathbb{R})}\|\bar{P}w\|_{L^2(\mathbb{R}^{n-1}_y)}\leqslant C\|w\|_{L^2(\mathbb{R}^n)},\tag{3.25}$$

where C is a constant depending only on c_0, the L^1 and L^2 norm of ϕ', ϕ^* and J. Taking the derivative with respect to ξ in Eq. (3.20) and projecting onto $QL^2(\mathbb{R}^n)$, for $v=w_\xi$, we get

$$v^\perp_t=\mathcal{L}v^\perp+\mathcal{B}v^\perp+Q\bigl(f''(\phi)\phi'w\bigr)+Q(Qh)_\xi.\tag{3.26}$$

Since $\|(f''(\phi)\phi'w)\|_{L^2(\mathbb{R}^n)}\leqslant C(\|\phi'\|_{H^1(\mathbb{R})})\|w\|_{L^2(\mathbb{R}^n)}$, from estimate (3.24), we see that $\tilde{h}=f''(\phi)\phi'w+(Qh)_\xi$ satisfies same assumption as h for the case $m=0$. Therefore we can repeat the procedure for the estimate of $\|w\|_{L^2(\mathbb{R}^n)}$ and get

$$\begin{aligned}\|Qw_\xi\|_{L^2(\mathbb{R}^n)}\leqslant{}& C\bigl(\|\phi'\|_{H^1(\mathbb{R})},\|\phi^*\|_{H^1(\mathbb{R})}\bigr)(1+t)^{-\alpha}\\ &\times\sup_{\tau\geqslant 0}(1+\tau)^\alpha\|h\|_{H^1(\mathbb{R}^n)}(\tau).\end{aligned}\tag{3.27}$$

Combining (3.27) with (3.25), we get the estimate for $\|w_\xi\|_{L^2(\mathbb{R}^n)}$. A similar argument gives estimates for higher derivatives. □

Lemma 3.4. *Suppose $v_0(\xi,y)\in H^m(\mathbb{R}^n)$ and w is the unique solution of the following initial value problem*:

$$w_t=\mathcal{L}w+\mathcal{B}w\quad\textit{and}\quad w|_{t=0}=Qv_0.$$

Then there exist constants $\gamma > 0$ and $C > 0$ such that $\|w\|_{H^m(\mathbb{R}^n)} \leqslant Ce^{-\gamma t} \times \|Qv_0\|_{H^m(\mathbb{R}^n)}$ for all $t \geqslant 0$.

The proof of the lemma is similar to that of Lemma 3.3. We omit it.

Proof of Theorem 3.1. We try to solve (3.8) and (3.9) by a contraction principle. For this purpose, let $n \geqslant 4$, $m \geqslant n+1$ and let $\alpha = (n-1)/4$. Define

$$X_\alpha = \Big\{ v(t,\xi,y) \in H^m(\mathbb{R}^n) \text{ for all } t \geqslant 0\colon \sup_{t\geqslant 0}(1+t)^\alpha \|v\|_{H^m(\mathbb{R}^n)}(t) < \infty \Big\}.$$

We endow X_α with the norm $\|v\|_\alpha = \sup_{t\geqslant 0}(1+t)^\alpha \|v\|_{H^m(\mathbb{R}^n)}(t)$. Then X_α is a Banach space. For $w \in X_\alpha$, consider the following equations:

$$v_t^\perp = \mathcal{L}v^\perp + \mathcal{B}v^\perp + QR(w,\phi), \tag{3.28}$$

$$p_t = \mathcal{B}p + \bar{P}R(w,\phi), \tag{3.29}$$

with the initial data $v^\perp|_{t=0} = Qv_0$ and $p|_{t=0} = \bar{P}v_0$. Let $v = v^\perp + \phi' p$ and define an operator $\mathcal{K}$ on X_α by $\mathcal{K}w = v$. For $\delta > 0$, let $B_\delta = \{w \in X_\alpha\colon \|w\|_\alpha \leqslant \delta\}$. We claim that $\mathcal{K}$ is a contraction map on B_δ if δ and $d_0 \equiv \|v_0\|_{H^m(\mathbb{R}^n)} + \|v_0\|_{L^1(\mathbb{R}^n)}$ are small enough.

First $|R(w,\phi)| \leqslant C(|w^3| + |w^2|)$ for some constant C depending only on $\|\phi'\|_{H^m}$. The Sobolev imbedding theorem implies that

$$\big\|R(w,\phi)\big\|_{H^m(\mathbb{R}^n)} \leqslant C\big(\|w\|^2_{H^m(\mathbb{R}^n)} + \|w\|^3_{H^m(\mathbb{R}^n)}\big)$$

for another constant C. Since $v^\perp = e^{(\mathcal{L}+B)t}Qv_0 + \int_0^t e^{(\mathcal{L}+B)(t-s)}QR(w,\phi)\,ds$, from Lemmas 3.3 and 3.4, we deduce that

$$\|v\|^\perp_{H^m(\mathbb{R}^n)} \leqslant \|Qv_0\|_{H^m(\mathbb{R}^n)}e^{-\gamma t} + C\big(\|w\|^2_{H^m(\mathbb{R}^n)} + \|w\|^3_{H^m(\mathbb{R}^n)}\big). \tag{3.30}$$

On the other hand, $p = e^{\mathcal{B}t}\bar{P}v_0 + \int_0^t e^{\mathcal{B}(t-s)}\bar{P}R(w,\phi)\,ds$. Therefore, by Lemma 3.2,

$$\begin{aligned}
\|p\|_{H^m(\mathbb{R}^{n-1})} &\leqslant Ce^{-\beta t}\|\bar{P}v_0\|_{H^m(\mathbb{R}^{n-1})} + C(1+t)^{-\alpha}\|\bar{P}v_0\|_{L^1(\mathbb{R}^{n-1})} \\
&\quad + \int_0^t e^{-\beta(t-s)}\big\|\bar{P}R(w,\phi)\big\|_{H^m(\mathbb{R}^{n-1})}\,ds \\
&\quad + C\int_0^t (1+t-s)^{-\alpha}\big\|\bar{P}R(w,\phi)\big\|_{L^1(\mathbb{R}^{n-1})}\,ds \\
&\leqslant C\Big\{ e^{-\beta t}\|v_0\|_{H^m(\mathbb{R}^n)} + (1+t)^{-\alpha}\|v_0\|_{L^1(\mathbb{R}^n)} \\
&\quad + (1+t)^{-2\alpha}\big(\|w\|^2_\alpha + \|w\|^3_\alpha\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^t (1+t-s)^{-\alpha}(1+s)^{-2\alpha}\,ds\,\big(\|w\|_\alpha^2 + \|w\|_\alpha^3\big)\bigg\} \\
&\leqslant C\big(\|v_0\|_{H^m(\mathbb{R}^n)} + \|v_0\|_{L^1(\mathbb{R}^n)} + \|w\|_\alpha^2 + \|w\|_\alpha^3\big) \\
&\quad \times (1+t)^{-\alpha},
\end{aligned}
\tag{3.31}
$$

where we have used the fact $\int_0^t (1+t-s)^{-\alpha}(1+s)^{-2\alpha}\,ds \leqslant C(1+t)^{-\alpha}$ (see [9] or [6]). Combining (3.31) with (3.30), we have

$$
\|v\|_\alpha \leqslant C\big(\|v_0\|_{H^m(\mathbb{R}^n)} + \|v_0\|_{L^1(\mathbb{R}^n)} + \|w\|_\alpha^2 + \|w\|_\alpha^3\big), \tag{3.32}
$$

where C is a constant only depending on $\|\phi'\|_{H^m(\mathbb{R})}$. Hence, $v \in B_\delta$ if δ and d_0 are small enough. Similarly, for $w_1, w_2 \in X_\alpha$, we can deduce that

$$
\|v_1 - v_2\|_\alpha \leqslant C\big(\|w_1\|_\alpha + \|w_1\|_\alpha^2 + \|w_2\|_\alpha + \|w_2\|_\alpha^2\big)\|w_1 - w_2\|_\alpha, \tag{3.33}
$$

where $v_i = \mathcal{K}w_i$ for $i = 1, 2$. From (3.32) and (3.33), we deduce that $\mathcal{K}$ is a contraction on $B\delta$ if d_0 and δ are small enough. Therefore (3.8) and (3.9) have a unique solution v and $\|v\|_{H^m(\mathbb{R}^n)} \leqslant C(1+t)^{-\alpha}$. That finishes the proof of Theorem 3.1. □

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